Semiclassical theory of ballistic transport through chaotic cavities with spin-orbit interaction

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We investigate the influence of spin-orbit interaction on ballistic transport through chaotic cavities by using semiclassical methods. Our approach is based on the Landauer formalism and the Fisher-Lee relations, appropriately generalized to spin-orbit interaction, and a semiclassical representation of Green functions. We calculate conductance coefficients by exploiting ergodicity and mixing of suitably combined classical spin-orbit dynamics, and making use of the Sieber-Richter method and its most recent extensions. That way we obtain weak anti-localization and confirm previous results obtained in the symplectic ensemble of Random Matrix Theory.

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I. INTRODUCTION

Ballistic transport through chaotic cavities realized as quantum dots in semiconductor heterostructures has been a central issue in mesoscopic physics for many years. The universal transport properties observed in this context can be described on a phenomenological level by random matrix theory[1] (RMT). The same applies to disordered systems, where averages over impurities can be shown to be equivalent to random matrix averages. This not being possible for individual, clean cavities, theoretical explanations of the RMT-connection have been provided making use of semiclassical methods, which are based on the Landauer formalism[2] and semiclassical representations of Green functions. This approach[3] leads to questions that are closely analogous to problems arising in semiclassical explanations of universal spectral correlations in classically chaotic quantum systems. Recent progress in the latter context is based on the seminal work of Sieber and Richter[4] and its extensions[5,6,7]. This method has been adapted[8,9,10] to be able to successfully explain conductance coefficients, including the effect of weak localization, i.e., a decrease of conductance at zero magnetic field. Further studies have been devoted to analyses of the universality of conductance fluctuations[10,11,12], and of shot noise[10,13,14]. (For an overview see, e.g., Ref. 10).

In the work mentioned transport properties were considered for ballistic, non-relativistic electrons, neglecting their spin. In the emerging field of semiconductor based spin electronics[15] (spintronics), however, one requires an efficient control of the spin dynamics associated with electrons in non-magnetic semiconductors. This purpose calls for an inclusion of spin-orbit interactions into studies of transport properties. In contrast to previous theories neglecting the spin, here one would expect appropriate classical spin-orbit dynamics to produce weak anti-localization, i.e., an enhancement of the conductance at zero magnetic field. This prediction is also obtained on the phenomenological level provided by RMT, where a half-integer spin requires the symplectic, as opposed to the orthogonal, circular ensemble. On this ground one expects universal conductance fluctuations and shot noise also to be affected by the presence of spin-orbit interactions[1,16]. A first semiclassical approach[17] to these questions employs the semiclassical representation of the Green function in spin-orbit coupling systems derived in Ref. 18 and considers the first order of the semiclassical Sieber-Richter expansion. It, moreover, assumes a randomization of spin states, which is shown to be responsible for weak anti-localization.

In this paper our goal is to extend the results of Ref. 17 to all orders of the Sieber-Richter expansion, and to base the semiclassical estimates entirely on dynamical properties of suitably combined classical spin-orbit dynamics[19]. These then replace the randomization hypothesis of spin states made in the analytic part of Ref. 17. In order to determine the spin contribution to transmission amplitudes we closely follow an analogous calculation introduced in the context of semiclassical explanations of spectral correlations in quantum graphs with spin-orbit couplings[20,21]. We also comment on shot noise and on the variance of conductance fluctuations.

As our model we consider a two dimensional cavity with two straight, semi-infinite leads with hard walls. Apart from boundary reflections, particles with mass m, charge e, and spin s move freely within the leads and are subjected to a magnetic field and to spin-orbit interactions inside the cavity. Although the relevant case of electrons enforces the spin to be $s = 1/2$, we deliberately allow for general spin s. Below this will allow us to point out characteristic differences between integer and half-integer spin. The Hamiltonian governing the dynamics in the cavity reads

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\right)^2 + \hat{\mathbf{s}}\cdot\mathbf{C}(\hat{\mathbf{x}},\hat{\mathbf{p}}) \ . \tag{1}$$

Here $\mathbf{A}$ is the vector potential for an external magnetic field and $\mathbf{C}$ contains all couplings of the translational degrees of freedom to the spin operator $\hat{\mathbf{s}}$. This may include Zeeman-, spin-orbit, Rashba-, or Dresselhaus-type couplings. Moreover, in order to model the hard walls we require Dirichlet conditions at the boundaries of the cavity and of the leads.

The paper is organized as follows: Section II is devoted to a generalization of the Landauer formalism and the Fisher-Lee relations to systems with spin-orbit interaction. Then we

present semiclassical representations of S-matrix elements in that case. In Section III we first introduce ergodicity and mixing conditions that include a classical spin-orbit interaction. This is followed by our calculation of the conductance in two ways: in the configuration-space and in the phase-space approach. In Sections IV and V we then outline how our approach can be extended to calculate shot noise and conductance fluctuations, respectively. An Appendix contains a calculation whose result is central to the phase-space approach employed in Section III.

II. PRELIMINARIES

We follow the usual approach to obtain semiclassical approximations to transmission by employing the Landauer formalism[2] and introducing semiclassical representations for Green functions. In the absence of spin-orbit interactions this procedure is well established[22,23,24]. Here we briefly describe the extensions required by the presence of spin-orbit interactions (see also Ref. 17).

A. Landauer formalism with spin

The Landauer formalism provides a link between conductance coefficients, as defined through

$$I_n = \sum_m g_{nm} V_m \ , \tag{2}$$

and S-matrix elements. In (2) the indices label the leads, V_m is the voltage applied at lead m and I_n is the current through lead n. Here the number of leads may be arbitrary. An S-matrix element $S^{nm}_{\alpha_n\alpha'_m}$ is defined as the transition amplitude between an asymptotic incoming state in the lead m, characterized by the collection α'_m of its quantum numbers, to an asymptotic outgoing state in the lead n, accordingly characterized by α_n.

In Refs. 23,24 the Landauer formalism was derived from the Schrödinger equation in linear response theory, making use of an appropriate Kubo-Greenwood formula. We first remark that an inclusion of spin, interacting with the translational degrees of freedom via a Zeeman, spin-orbit, Rashba, or Dresselhaus coupling, into this method causes no problems. Although the current density is modified, its conservation in the form required for the Kubo-Greenwood expression of the conductivity to hold is indeed guaranteed. On then obtains for transmission (i.e. $m \neq n$),

$$g_{nm} = -\frac{e^2}{h}\int_0^\infty dE\, f'_\beta(E) \sum_{\alpha_n,\alpha'_m} \left| S^{nm}_{\alpha_n\alpha'_m} \right|^2 \ , \tag{3}$$

and for reflection (i.e. $m = n$),

$$g_{nn} = \frac{e^2}{h}\int_0^\infty dE\, f'_\beta(E) \left((2s+1)N_n - \sum_{\alpha_n,\alpha'_n} \left| S^{nn}_{\alpha_n\alpha'_n} \right|^2 \right) \ . \tag{4}$$

Here N_n is the number of open channels in the lead n (without spin degeneracy) at energy E, and $f_\beta(E)$ denotes the Fermi distribution function at inverse temperature β. Of course, this requires the spin quantum number s to be half-integer.

In a next step S-matrix elements have to be related to Green functions $G(\mathbf{x},\mathbf{x}',E)$. These satisfy the equations

$$\left(\tfrac{1}{2m}\left(\hat{\mathbf{p}} - \tfrac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\right)^2 + \hat{\mathbf{s}}\cdot\mathbf{C}(\hat{\mathbf{x}},\hat{\mathbf{p}}) - E \right) G(\mathbf{x},\mathbf{x}',E) = \delta(\mathbf{x}-\mathbf{x}') \tag{5}$$

and

$$\left(\tfrac{1}{2m}\left(\hat{\mathbf{p}}' + \tfrac{e}{c}\mathbf{A}(\hat{\mathbf{x}}')\right)^2 - E \right) G(\mathbf{x},\mathbf{x}',E) + \mathbf{C}^*(\hat{\mathbf{x}}',\hat{\mathbf{p}}')G(\mathbf{x},\mathbf{x}',E)\hat{\mathbf{s}} = \delta(\mathbf{x}-\mathbf{x}') \ . \tag{6}$$

The unusual form of the second equation is dictated by the fact that $G(\mathbf{x},\mathbf{x}',E)$ is a hermitian $(2s+1)\times(2s+1)$ matrix in spin space. In the following we will always choose advanced Green functions, fully characterized by Eqs. (5) and (6) as well as the condition that, asymptotically in the leads, they contain only outgoing contributions.

As in the case without spin[23,24] one can then express the S-matrix elements in terms of the (advanced) Green function. Up to a global phase factor, for $m \neq n$ this yields

$$S^{nm}_{\alpha_n\alpha'_m} = \frac{2\hbar^2}{im}\sqrt{\frac{k_{a_n}k_{a'_m}}{W_m W_n}} \int_0^{W_n} dy_n \int_0^{W_m} dy'_m \sin\left(\frac{a_n\pi y_n}{W_n}\right) \times \sin\left(\frac{a'_m\pi y'_m}{W_m}\right) G_{\sigma\sigma'}(\mathbf{x}_n,\mathbf{x}'_m,E) \ , \tag{7}$$

and for $m = n$

$$S^{nn}_{\alpha_n\alpha'_n} = \frac{2\hbar^2}{im}\frac{\sqrt{k_{a_n}k_{a'_n}}}{W_n} \int_0^{W_n} dy_n \int_0^{W_n} dy'_n \sin\left(\frac{a_n\pi y_n}{W_n}\right) \times \sin\left(\frac{a'_n\pi y'_n}{W_n}\right) G_{\sigma\sigma'}(\mathbf{x}_n,\mathbf{x}'_n,E) + \delta_{\alpha_n\alpha'_n} \ . \tag{8}$$

Here we have introduced coordinates $\mathbf{x}_n = (x_n, y_n)$, where $x_n \geq 0$ is a longitudinal, outward running coordinate in the lead n and $0 \leq y_n \leq W_n$ is the corresponding transversal coordinate (see also Figure 1). The transversal quantum number is $a_n = 1,\ldots,N_n$ with associated wave number $k_{a_n} = \sqrt{2mE/\hbar^2 - a_n^2\pi^2/W_n^2}$. The number N_n of open transversal channels then is the largest integer a_n that leaves the wave number real. Moreover, $\sigma = -s,\ldots,s$ is a spin index such that altogether $\alpha_n = (E, a_n, \sigma)$.

We remark that in Eqs. (7) and (8) the points $\mathbf{x}_n, \mathbf{x}'_m$ can be chosen anywhere in the respective leads. For later convenience we take them on the connection of the leads to the cavity, i.e., with $x_n = 0 = x'_m$.

B. Semiclassical Green function and transmission amplitudes

In order to proceed further, one requires a semiclassical representation for the Green function defined in Eqs. (5) and (6). In Ref. 18 this was achieved through an asymptotic expansion in powers of Planck's constant $\hbar$ for the quantum propagator generated by the Hamiltonian (1) which yielded, after a

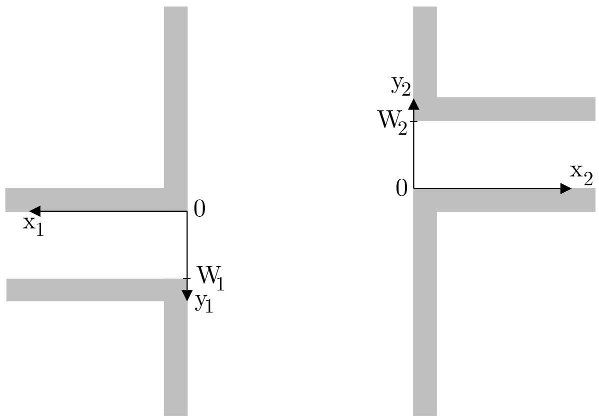

Figure 1: Sketch of the geometry

Fourier transformation, a respective semiclassical expansion for the Green function. The range of validity of this procedure follows from the observation that, since the spin operator $\hat{\mathbf{s}}$ is linear in $\hbar$, the energy scale of the spin-orbit interaction term becomes small as compared to the kinetic term in the limit $\hbar \to 0$. This condition is equivalent to the spin-precession length being large compared to the Fermi wavelength. In semiconductor heterostructures this requirement is usually fulfilled.

The semiclassical representation for the Green function obtained in Ref. 18 reads

$$G(\mathbf{x},\mathbf{x}',E) \sim \sum_{\gamma(\mathbf{x},\mathbf{x}')} A_\gamma(\mathbf{x},\mathbf{x}',E)\,\exp\big((i/\hbar)S_\gamma(\mathbf{x},\mathbf{x}',E)\big) \ , \quad (9)$$

as $\hbar \to 0$. The sum extends over all classical trajectories $\gamma(\mathbf{x},\mathbf{x}')$ generated by the classical Hamiltonian

$$H_0(\mathbf{x},\mathbf{p}) = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x})\right)^2 \quad (10)$$

(plus reflections from hard walls) that run from $\mathbf{x}'$ to $\mathbf{x}$ at energy E. Choosing $(\mathbf{x},\mathbf{x}') = (\mathbf{x}_n,\mathbf{x}'_m)$ as in (7), (8), the relevant trajectories are those that enter the cavity at lead m and leave through lead n. Moreover, $S_\gamma(\mathbf{x},\mathbf{x}',E)$ is the classical action of the trajectory, and the leading order of the amplitude $A_\gamma(\mathbf{x},\mathbf{x}',E)$ reads

$$A_\gamma(\mathbf{x},\mathbf{x}',E) = \frac{e^{-i\frac{\pi}{2}\nu_\gamma}}{i\hbar\sqrt{2\pi i\hbar}}\sqrt{C_\gamma}D_\gamma(\mathbf{x}',\mathbf{p}',t)\big(1+O(\hbar)\big) \ . \quad (11)$$

Here ν_γ is a Maslov index of the trajectory γ, and

$$C_\gamma := \left|\det\begin{pmatrix} \frac{\partial^2 S_\gamma}{\partial\mathbf{x}\partial\mathbf{x}'} & \frac{\partial^2 S_\gamma}{\partial\mathbf{x}\partial E} \\ \frac{\partial^2 S_\gamma}{\partial\mathbf{x}'\partial E} & \frac{\partial^2 S_\gamma}{\partial E^2}\end{pmatrix}\right| \ . \quad (12)$$

The contribution of the spin is, in leading semiclassical order, completely contained in the spin-transport matrix $D_\gamma(\mathbf{x}',\mathbf{p}',t)$. This is the spin-$s$ representation of the spin propagator $d_\gamma(\mathbf{x}',\mathbf{p}',t)$, which is defined as a solution of the equation

$$\frac{d}{dt}d_\gamma(\mathbf{x}',\mathbf{p}',t) + \frac{i}{2}\mathbf{C}\left(\mathbf{X}(t),\mathbf{P}(t)\right)\cdot\boldsymbol{\sigma}\,d_\gamma(\mathbf{x}',\mathbf{p}',t) = 0 \quad (13)$$

with initial condition $d_\gamma(\mathbf{x}',\mathbf{p}',0) = 1$. Here $(\mathbf{X}(t),\mathbf{P}(t))$ is the point in phase space of the classical trajectory γ at time t. Its initial point at time $t=0$ is $(\mathbf{x}',\mathbf{p}')$. Moreover, $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices. Therefore, d_γ is an SU(2)-matrix that can be seen as a propagator for the spin along the classical trajectory γ.

Upon dividing the trajectory γ into two pieces γ_1 and γ_2, such that $t = t_1 + t_2$, the spin propagator is clearly multiplicative. Since D_γ arises from a group representation it inherits this multiplicative property from the propagator, i.e.,

$$D_\gamma(\mathbf{x}',\mathbf{p}',t_1+t_2) = D_{\gamma_2}(\mathbf{X}(t_1),\mathbf{P}(t_1),t_2)\,D_{\gamma_1}(\mathbf{x}',\mathbf{p}',t_1) \ . \quad (14)$$

This relation will be used extensively in Section III.

In order to obtain a semiclassical representation of transmission amplitudes we insert the expression (9) into equation (7). Then the integrals over y and y', respectively, are evaluated, asymptotically as $\hbar \to 0$, with the method of stationary phase. In this context we stress the following important observation: The number of accessible transversal states (including spin) in the n-th lead is $(2s+1)N_n = (2s+1)[\sqrt{2mE}W_n/(\pi\hbar)]$, where $[x]$ denotes the integer part of $x \in \mathbb{R}$. We choose the widths W_n of the leads to formally shrink proportionally to $\hbar$ in this limit (compare also Ref. 14) and hence set $W_n = \tilde{W}_n\hbar$, to the effect that the sin-factors in Eqs. (7) and (8) contribute rapidly oscillating phases. These have to be taken into account when determining stationary points of the total phases in the integrals. The condition of stationary phase hence imposes the following restrictions on the transversal momenta,

$$p'_y = -\frac{\partial S_\gamma}{\partial y'_m} = \pm\frac{a'_m\pi}{\tilde{W}_m} \quad (15)$$

and

$$p_y = \frac{\partial S_\gamma}{\partial y_n} = \mp\frac{a_n\pi}{\tilde{W}_n} \ , \quad (16)$$

upon entry and exit, respectively, of the trajectories. If the points of entry and exit are free of magnetic fields, and thus $\mathbf{p} = m\dot{\mathbf{x}}$ at these points, one can characterize the trajectories in terms of the angles θ and θ', under which they enter and leave the cavity with respect to the longitudinal directions of the leads (see also Figure 2). These angles are related to the transversal momenta (15) and (16) through $\sin\theta = p_y/\sqrt{2mE}$ and $\sin\theta' = p'_y/\sqrt{2mE}$. If one wished to keep the widths of the openings fixed, however, the method of stationary phase would enforce the conditions $p'_y = 0 = p_y$ upon the trajectories, thus leading to different semiclassical expressions than the ones we use henceforth.

Collecting now all terms that emerge in the stationary phase calculation finally leads to the following leading semiclassical contribution to the S-matrix elements,

$$S^{nm}_{\alpha_n\alpha'_m} \sim \sum_{\gamma(\theta,\theta')} B_{\gamma(\theta,\theta')}\,D^{\sigma\sigma'}_{\gamma(\theta,\theta')}\,\exp\big((i/\hbar)\,S_{\gamma(\theta,\theta')}\big) \ , \quad (17)$$

where the sum extends over all trajectories that run from lead m through the cavity to lead n and are characterized by the conditions (15), (16), expressed in terms of the angles of entry

and exit. The explicit form of the factor $B_{\gamma(\theta,\theta')}$ is the same as if there were no spin present[23],

$$B_{\gamma(\theta,\theta')} = \sqrt{\frac{i\pi\hbar}{2W_m W_n}} \frac{\operatorname{sgn}(\pm a'_m)\operatorname{sgn}(\pm a_n)}{|\cos\theta\cos\theta' M^{21}_{\gamma(\theta,\theta')}|^{1/2}} \times \exp\left(i\pi\left(\frac{\pm a'_m y'_m}{W_m} + \frac{\pm a_n y_n}{W_n} - \frac{1}{2}\mu_{\gamma(\theta,\theta')}\right)\right) . \tag{18}$$

Here $M^{21}_{\gamma(\theta,\theta')}$ is an element of the monodromy matrix of $\gamma(\theta,\theta')$ that arises from the matrix appearing in (12) by a restriction to the phase space directions transversal to the trajectory. Furthermore, $\mu_{\gamma(\theta,\theta')}$ is a modified Maslov index that contains the index $\nu_{\gamma(\theta,\theta')}$ from Eq. (11) and additional phases resulting from the stationary phase calculation of the integrals over y_n and y'_m.

The above result (17) primarily refers to transmission amplitudes ($n \neq m$), but can be carried over to the case of reflection ($n = m$). The reason for this is that the additional term $\delta_{\alpha_n\alpha'_n}$ in (8) is canceled by the contribution of direct trajectories in the opening of the lead n that never enter the cavity[3].

The ultimate goal being a semiclassical calculation of the conductance coefficients (3) and (4), one therefore requires the evaluation of double sums

$$\left|S^{nm}_{\alpha_n\alpha'_m}\right|^2 \sim \sum_{\gamma(\theta,\theta')}\sum_{\gamma'(\theta,\theta')} B_\gamma B^*_{\gamma'} D^{\sigma\sigma'}_\gamma D^{\sigma\sigma'\,*}_{\gamma'} \times \exp\left((i/\hbar)\left(S_\gamma - S_{\gamma'}\right)\right) \tag{19}$$

over classical trajectories. This will be the task for the rest of this paper.

To simplify the calculations, from now on we restrict our attention to the case of two leads. With an incoming wave in the lead $m = 1$ we are thus dealing with the transmission coefficient g_{21} and the reflection coefficient g_{11}. To this end we will determine the transmission matrix S^{21} and the reflection matrix S^{11}, leading to the transmission and reflection coefficient

$$\mathcal{T} = \sum_{\alpha_2,\alpha'_1} \left|S^{21}_{\alpha_2\alpha'_1}\right|^2 , \quad \mathcal{R} = \sum_{\alpha_2,\alpha'_2} \left|S^{22}_{\alpha_2\alpha'_2}\right|^2 , \tag{20}$$

respectively. Hence, at zero temperature the current through lead 2 is

$$I_2 = \frac{e^2}{h}\left(\mathcal{T}\,V_1 + (\mathcal{R} - (2s+1)N_2)V_2\right) , \tag{21}$$

where $\mathcal{T}$ and $\mathcal{R}$ are taken at the Fermi energy E_F. Together with the condition $g_{21} + g_{22} = 0$, expressing that equal voltages at both leads produce no current, this yields the relation

$$I_2 = \frac{e^2}{h}\mathcal{T}\,(V_1 - V_2) . \tag{22}$$

III. SEMICLASSICAL CALCULATION OF CONDUCTIVITY COEFFICIENTS

The calculation of the double sum (19) over classical trajectories requires input from dynamical properties of the associated classical system. With spin-orbit interactions present, one therefore first has to identify an appropriate classical system. Moreover, ergodic properties of the classical system imply necessary ingredients for the further calculation. The diagonal contribution to the double sum is evaluated with a sum rule[8,17], whereas the non-diagonal terms are evaluated following the Sieber-Richter method[4,8,14,17].

A. Classical spin-orbit dynamics

The classical dynamics that enter the semiclassical representation (9) consist of two parts[18]: the motion of the point particle generated by the Hamiltonian (10), including elastic reflections from hard walls, and the spin that is driven by this motion according to (13). These contributions can be combined into a single dynamics on a spin-orbit phase space[19]. The relevant classical trajectory is $(\mathbf{X}(t),\mathbf{P}(t),g(t))$, with initial condition $(\mathbf{x}',\mathbf{p}',g')$ at $t = 0$. Here $g \in \mathrm{SU}(2)$ and $g(t) = d_\gamma(\mathbf{x}',\mathbf{p}',t)g$ provides the spin part of the combined motion. We remark that this description of spin appears quantum mechanical. However, by passing to expectation values of the spin operator $d^\dagger_\gamma \frac{1}{2}\boldsymbol{\sigma} d_\gamma$ in normalized spin states χ (Heisenberg picture), the spin variable becomes a unit vector $\langle\chi, d^\dagger_\gamma \frac{1}{2}\boldsymbol{\sigma} d_\gamma\chi\rangle$. Hence the spin part of the combined phase space is a unit sphere. The two views of the spin motion, either on $\mathrm{SU}(2)$ or on a unit sphere, are in fact equivalent[18]. In both cases we will therefore speak of classical spin-orbit dynamics.

Ergodicity is a concept developed for closed systems. It can, however, be suitably extended to open systems of the kind under consideration here. To this end one divides the configuration space Q of the device into a closed part Q_c, consisting of the cavity with the leads truncated and the openings closed, plus the infinite leads. From now on we suppose the shape of the closed part to form a chaotic billiard, ensuring ergodicity of the motion inside the cavity. Then $\rho(t)$ is the probability for a typical trajectory to stay within the cavity at least up to time t. For large times,

$$\rho(t) \sim \exp(-t/\tau) , \quad t \to \infty , \tag{23}$$

with inverse dwell time

$$\frac{1}{\tau} = \frac{\hbar}{mA}(N_1 + N_2) , \tag{24}$$

in which A denotes the area of the closed part Q_c. For the associated part of phase space we also introduce the volume

$$\Sigma(E) = \int_{Q_c} d^2x \int_{\mathbb{R}^2} d^2p\, \delta\left(E - H_0(\mathbf{x},\mathbf{p})\right) = 2\pi mA \tag{25}$$

of the energy shell. This expression has no integration over the spin part, since the Hamiltonian is independent thereof,

and an integration over SU(2) with respect to Haar measure dg yields one.

For the open system the concept of ergodicity has to be modifed in that the possibility of a trajectory to leave the cavity must be taken into account. When the motion inside the cavity is ergodic this leads to the following relation between phase-space averages and time averages over typical spin-orbit trajectories,

$$\left\langle \int_0^T dt\, f\left(\mathbf{X}(t),\mathbf{P}(t),g(t)\right)\right\rangle \sim \frac{1}{\Sigma(E)}\int_0^T dt\,\rho(t)\times \int_{Q_c} d^2x \int_{\mathbb{R}^2} d^2p \int_{SU(2)} dg\, f(\mathbf{x},\mathbf{p},g)\,\delta\left(E-H_0(\mathbf{x},\mathbf{p})\right) \ , \quad (26)$$

as $T\to\infty$. Here f is an arbitrary function on the combined phase space, and $\langle\ldots\rangle$ denotes an average over initial conditions. This relation, which properly reflects the chaotic nature of the combined classical spin-orbit motion, provides the basis for the further use of dynamical properties in the calculation of the sum (19) over classical trajectories.

The stronger mixing property, which we also assume to hold henceforth, means that correlations of two observables f and h decay, i.e.,

$$\lim_{t\to\infty}\int_{Q_c} d^2x \int_{\mathbb{R}^2} d^2p \int_{SU(2)} dg\; h\left(\mathbf{X}(t),\mathbf{P}(t),g(t)\right) f\left(\mathbf{x},\mathbf{p},g\right)\delta\left(E-H_0(\mathbf{x},\mathbf{p})\right)$$
$$=\frac{1}{\Sigma(E)}\int_{Q_c} d^2x \int_{\mathbb{R}^2} d^2p \int_{SU(2)} dg\; h(\mathbf{x},\mathbf{p},g)\delta\left(E-H_0(\mathbf{x},\mathbf{p})\right)\;\int_{Q_c} d^2x' \int_{\mathbb{R}^2} d^2p' \int_{SU(2)} dg'\, f(\mathbf{x}',\mathbf{p}',g')\delta\left(E-H_0(\mathbf{x}',\mathbf{p}')\right)\ . \quad (27)$$

B. Transmission and reflection coefficients in the configuration-space approach

In a first step we calculate the leading semiclassical contribution to transmission and reflection coefficients from equation (19), averaged over a small energy window, by using the configuration-space approach. Such a calculation has been performed previously[17], however, with a sum rule that only takes the particle motion into account. The spin contribution was built in subsequently, assuming that traces of products of spin-transport matrices can be replaced by certain averages. Here we reproduce the result obtained in Ref. 17 by using a sum rule for the complete spin-orbit dynamics that follows from (26). Thus we base the assumptions made in Ref. 17 on a firm dynamical ground.

As $\hbar\to 0$ the terms in the double sum (19) are highly oscillatory, except for contributions with $S_\gamma = S_{\gamma'}$. Generically, if no symmetries are present, this only occurs for the diagonal $\gamma'=\gamma$. In the event that time-reversal invariance is not broken, however, the time-reversed trajectory γ^{-1} has the same action as γ. Of course, γ^{-1} is only among the trajectories to be summed over in the case of reflection ($n=1=m$) when, moreover, $\theta=\theta'$; i.e., only for $S^{11}_{\alpha_1\alpha'_1}$ with $a_1=a'_1$. All further terms are oscillatory, with a decreasing importance of their contribution, after averaging over an energy window, when the action differences increase. Below we calculate the two leading contributions to the quantity

$$\sum_{\sigma,\sigma'=-s}^{s}\left|S^{nm}_{\alpha_n\alpha'_m}\right|^2 \sim \sum_{\gamma,\gamma'} B_\gamma B^*_{\gamma'}\,\mathrm{Tr}\,(D_\gamma D^\dagger_{\gamma'})\exp\left((i/\hbar)\left(S_\gamma - S_{\gamma'}\right)\right) \quad (28)$$

for systems with time-reversal invariance: (i) the diagonal contribution in which the sum over γ' is restricted to $\gamma'=\gamma$ (for transmission) or $\gamma'=\gamma^{\pm1}$ (for reflection), and (ii) the one-loop contribution in which the sums over γ and γ' are confined to so-called Sieber-Richter pairs (see also Ref. 17).

Due to the unitarity of the spin-transport matrices, in the diagonal case terms with $\gamma'=\gamma$ yield a spin contribution of $\mathrm{Tr}\,(D_\gamma D^\dagger_\gamma)=2s+1$. Thus, the diagonal contribution to (28) can immediately be obtained from the respective result without spin[3,8],

$$\left\langle \sum_{\sigma,\sigma'=-s}^{s}\left|S^{nm}_{\alpha_n\alpha'_m}\right|^2_{\mathrm{diag}}\right\rangle_{\Delta E} \sim \frac{2s+1}{N_1+N_2}\ . \quad (29)$$

In the case of reflection ($n=1=m$) with $a_1=a'_1$ an additional diagonal contribution arises from the terms with $\gamma'=\gamma^{-1}$, if time-reversal invariance is unbroken. Its spin contribution is $\mathrm{Tr}\,(D_\gamma D^\dagger_{\gamma^{-1}})=\mathrm{Tr}\,(D^2_\gamma)$. One hence requires a suitable sum rule that incorporates the combined classical spin-orbit motion. For this purpose we choose the function

$$\begin{aligned} f&\left(\mathbf{X}(t),\mathbf{P}(t),g(t)\right)\\ &=\frac{1}{m}\delta(\vartheta(t)-\theta)\,\delta(x(t))\left(\Theta(y(t))-\Theta(y(t)-W_1)\right)\\ &\quad\times\,\mathrm{Tr}\left(\pi_s(g(t)g(0)^{-1})\right)^2 \end{aligned} \quad (30)$$

in (26). Here $\pi_s(g)$ denotes the spin-s representation of $g\in \mathrm{SU}(2)$, ϑ is the angular variable in planar polar coordinates for $\mathbf{p}$ and $\Theta(y)$ is a Heavyside step function. An evaluation of (26) with the function (30) then leads to the sum rule (as $T\to\infty$)

$$\sum_{\gamma,T_\gamma\le T}\left|B_\gamma\right|^2\mathrm{Tr}\,(D^2_\gamma)\sim\frac{\pi}{2\tilde{W}_1}\frac{(-1)^{2s}}{2\pi mA}\int_0^T dt\,\rho(t)\ . \quad (31)$$

After an average over a small window in energy this, together with (29), finally yields the semiclassical result

$$\left\langle \sum_{\sigma,\sigma'=-s}^{s}\left|S^{11}_{\alpha_1\alpha'_1}\right|^2_{\mathrm{diag}}\right\rangle_{\Delta E}\sim\frac{2s+1+(-1)^{2s}\delta_{a_1a'_1}}{N_1+N_2} \quad (32)$$

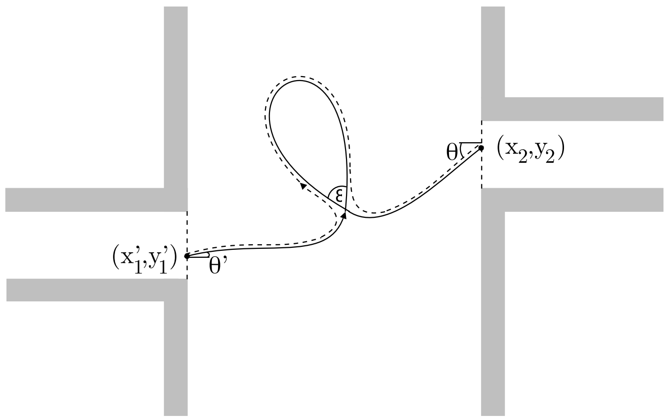

Figure 2: A Sieber-Richter pair of trajectories

for the diagonal contribution to (28). For $s = 1/2$ the right-hand side is $1/(N_1 + N_2)$.

Sieber-Richter pairs of trajectories are characterized by the fact that one trajectory possesses a self-crossing with a small crossing angle ε, thus forming a loop. The partner trajectory then looks like the former one cut open at the self-crossing, but with the loop direction reversed and then glued together, such that the self-crossing is replaced by an almost-crossing, see Figure 2. In principle, the trajectories in such pairs can have an arbitrary number of self-crossings, but the magnitude of their contributions to (28) decreases with increasing numbers of places in which self-crossings are paired with almost-crossings. The most important ('one-loop') contribution comes from pairs which differ in one crossing. In order to calculate the one-loop contribution one requires the distribution of the crossing angles ε for pairs of trajectories with loops of duration T,

$$P_S(\varepsilon,T) = \frac{1}{\Sigma(E)} \int_{Q_c} d^2x' \int_{\mathbb{R}^2} d^2p' \int_{T_{min}(\varepsilon)}^{T} dt_l \, p_S(\varepsilon,T,t_l) \; . \tag{33}$$

Here $p_S(\varepsilon,T,t_l)$ is a density of crossing angles defined as

$$\begin{aligned} p_S(\varepsilon,T,t_l) \; &= \; \int_0^{T-t_l} dt_s \, |J| \, \delta(E - H_0(\mathbf{P}(t_s))) \\ &\quad \times \mathrm{Tr}\left(\pi_s \left[g(t)(g(0))^{-1}\right]\right)^2 \delta(\varepsilon - \kappa(t_s,t_l)) \\ &\quad \times \delta(\mathbf{X}(t_s) - \mathbf{X}(t_s + t_l)) \; , \end{aligned} \tag{34}$$

where $\kappa(t_s,t_l)$ denotes the angle between the velocities $\mathbf{v}(t_s)$ and $\mathbf{v}(t_s + t_l)$. Given a crossing angle ε, the minimal duration for a loop to close is $T_{min}(\varepsilon)$. In chaotic systems this quantity behaves like $T_{min}(\varepsilon) = O(\log\varepsilon)$ as $\varepsilon \to 0$[4]. Furthermore,

$$\begin{aligned} |J| &= |\mathbf{v}(t_s) \times \mathbf{v}(t_s + t_l)| \\ &= |\mathbf{v}(t_s)| \, |\mathbf{v}(t_s + t_l)| \sin\kappa(t_s,t_l) \end{aligned} \tag{35}$$

is a Jacobian, and t_s, t_l denote the time along the trajectory up to the starting point of the loop and along the loop, respectively.

Assuming that the classical spin-orbit dynamics are not only ergodic, but also mixing, the distribution (33) can be calculated further. It can be identified as the left-hand side of an appropriate relation of the type (27). The right-hand side then yields, as $\varepsilon \to 0$,

$$P_S(\varepsilon,T) \sim \frac{(-1)^{2s}}{\pi A} \, \frac{2E}{m} \, \sin\varepsilon \left(\frac{T^2}{2} - T\,T_{min}(\varepsilon) + \frac{T_{min}^2(\varepsilon)}{2} \right) \; . \tag{36}$$

This expression differs from the respective one without spin that was obtained in Ref. 4 only by a factor $(-1)^{2s}$, i.e., a sign in the case $s = 1/2$. With this information at hand the one-loop contribution can be calculated as in the case without spin[8], finally yielding

$$\left\langle \sum_{\sigma,\sigma'=-s}^{s} \left| S^{21}_{\alpha_1\alpha'_2} \right|^2_{1-\mathrm{loop}} \right\rangle_{\Delta E} \sim -\frac{(-1)^{2s}}{(N_1+N_2)^2} \; . \tag{37}$$

This is in accordance with what has been obtained in Ref. 17.

C. Transmission coefficients in the phase-space approach

Higher orders in the 'loop-expansion' described above have been calculated previously for spectral form factors[6] as well as for conductance coefficients for systems without spin contributions[9]. The approach taken in these papers utilizes trajectories in classical phase space and identifies the pairs of self-crossings/almost-crossings in configuration space as pairs of trajectories with almost-crossings in phase space, which differ in the way they are connected at the (almost) crossings. This point of view opens the possibility for a classification of the trajectory pairs in terms of their encounters[6]. Here we follow this phase-space approach and amend the previous result[9] with the contribution of the spin-orbit coupling.

To be more precise, we consider trajectories that possess close self-encounters (in phase space), in which two or more short stretches of the trajectory are almost identical, possibly up to time reversal. These stretches are connected by long parts of the trajectory, which we call loops. We then form pairs (γ,γ') of such trajectories in which γ and γ' are almost identical (up to time reversal) along the loops, but differ from each other in the way the loops are connected in the encounter region. In order to quantify these encounters, we introduce a vector $\vec{v}$, whose l-th component, v_l, denotes the number of encounters with l stretches. Hence the total number of encounters is $V = \sum_{l\geq 2} v_l$, with a total of $L = \sum_{l\geq 2} l v_l$ stretches involved. In general, however, given a vector $\vec{v}$, there will be $N(\vec{v}) \geq 1$ different trajectory pairs associated with it. These may, e.g., differ in the order the loops connect the encounters, or in the relative directions, in which the encounter-stretches are traversed.

To reveal the phase-space structure of trajectory pairs and to compute their contributions to (19) one introduces Poincaré sections, which cut the trajectories into pieces. In order to adapt this cutting to the sequence of encounters and loops one chooses a Poincaré section in every of the V given encounter regions. We then denote by $t'_{\alpha,j}$, $j = 1,\ldots,l_\alpha$, $\alpha = 1,\ldots,V$ the

times at which the encounter stretches pierce this section, and by t^{α}_{enc}, the duration of the encounters. To this cutting of the trajectories corresponds the splitting

$$D_{\gamma} = D_{L+1} D_{L} ... D_{1} \tag{38}$$

of the spin-transport matrices which, with an obvious notation, follows from the composition rule (14). The spin transport along the partner trajectory then reads

$$D_{\gamma'} \approx D_{L+1} D^{\eta_L}_{k_L} ... D^{\eta_2}_{k_2} D_1 \ . \tag{39}$$

Here $\eta_j = \pm 1$, depending on the relative orientation of the trajectory between the $j-1$-st and the j-th cutting of γ and γ', respectively, through the Poincaré section. We notice that at this point time-reversal invariance enters crucially. Moreover, the indices k_j take care of the fact that in γ and γ' the loops may be traversed in different successions. Thus the spin-dependent weights in (28) for each pair of trajectories are approximately given by

$$\mathrm{Tr}\left(D_{\gamma} D^{\dagger}_{\gamma'}\right) \approx \mathrm{Tr}\left(D_L ... D_2 D^{\dagger \eta_2}_{k_2} \ldots D^{\dagger \eta_L}_{k_L}\right) \ . \tag{40}$$

The calculation of transmission amplitudes performed in Ref. 9 has now to be modified in that the expressions (40) must be included. To this end we recall the strategy devised in Refs. 6,9: For each encounter one introduces coordinates on the Poincaré section adapted to the piercing by the trajectories and the linear stability of the dynamics. In encounter α the coordinates $(s^{\alpha}_{j}, u^{\alpha}_{j})$, $j = 1, \ldots, l_{\alpha} - 1$, describe the separation of the $j+1$-st piercing from the j-th one along the stable and unstable manifolds, respectively, of the latter. The total of $L-V$ stable and unstable coordinates are then collected in the vectors $(\mathbf{s}, \mathbf{u})$. In these coordinates action differences of partner trajectories (approximately) read as

$$\Delta S = S_{\gamma} - S_{\gamma'} \approx \sum_{\alpha, j} s^{\alpha}_{j} u^{\alpha}_{j} \ . \tag{41}$$

Moreover, the requirement that encounters be close can then be expressed in terms of the condition $|s^{\alpha}_{j}|, |u^{\alpha}_{j}| \leq c$ with some constant c, which yields the duration of an encounter

$$t^{\alpha}_{enc} \sim \frac{1}{\lambda} \ln \frac{c^2}{\max_i \{|s_i|\} \max_j \{|u_j|\}}, \quad t^{\alpha}_{enc} \to \infty \ . \tag{42}$$

One then introduces a density $w^{\mathrm{spin}}_{T}(\mathbf{s}, \mathbf{u})$ of encounters, weighted with the spin contribution, for trajectories of duration T with a given encounter structure specified by the vector $\vec{v}$. In analogy to the case without spin[14] this leads to the following approximation,

$$\begin{aligned} \Big\langle \sum_{\gamma} \sum_{\vec{v}} N(\vec{v}) \int_{-c}^{c} \cdots \int_{-c}^{c} d^{L-V} u \, d^{L-V} s \, \exp\left((i/\hbar)\Delta S\right) \\ \times w^{\mathrm{spin}}_{T}(\mathbf{s}, \mathbf{u}) \, |B_{\gamma}|^2 \Big\rangle_{\Delta E} \ , \end{aligned} \tag{43}$$

to the quantity

$$T^{\mathrm{nd}}_{a_2 a'_1} := \Big\langle \sum_{\sigma, \sigma' = -s}^{s} \left| S^{21}_{\alpha_2 \alpha'_1} \right|^2 \Big\rangle_{\Delta E} - \frac{2s+1}{N_1 + N_2} \ . \tag{44}$$

After summing over all possible values of a_2, a'_1, this yields the non-diagonal contribution to the energy-averaged transmission amplitude $\mathcal{T}$, compare (20), (29).

The essential point now is to calculate the density $w^{\mathrm{spin}}_{T}(\mathbf{s}, \mathbf{u})$. In the case without spin-orbit interaction the corresponding expression $w_T(\mathbf{s}, \mathbf{u})$ was defined in Ref. 6 as a density of phase-space separations $\mathbf{s}$ and $\mathbf{u}$ similar to the density $P(\varepsilon, T)$ with respect to ε in the configuration-space approach. It was given as

$$\begin{aligned} w_T(\mathbf{s}, \mathbf{u}) \ = \ & \frac{1}{\Sigma(E)} \int_{Q_c} d^2 x' \int_{\mathbb{R}^2} d^2 p' \delta\left(E - H_0(\mathbf{x}', \mathbf{p}')\right) \\ & \times \int_0^{\infty} \prod_{j=1}^{L} dt_j \ \Theta\left(T - \sum_{\alpha=1}^{V} l_{\alpha} t^{\alpha}_{enc} - \sum_{j=1}^{L} t_j\right) \\ & \times \prod_{\alpha=1}^{V} \frac{1}{t^{\alpha}_{enc}} \left(\prod_{j=2}^{l_{\alpha}} \delta\left(\left(\mathbf{X}(t'_{\alpha j}), \mathbf{P}(t'_{\alpha j})\right) - z_{\alpha j}\right)\right) \ . \end{aligned} \tag{45}$$

The average in the first line is over all possible initial points of the trajectory. In the second line the integration extends over all loop durations t_j; their lengths are constrained by the theta function. In order to prevent over-counting[6], the product of all encounter durations t^{α}_{enc} is divided out. The last product guarantees that the position of the orbit at times when it pierces through the sections are fixed as $z_{\alpha j}$. This denotes the first point of the orbit in which it pierces through a certain section plus the separation thereof as specified by the coordinates $\mathbf{s}$ and $\mathbf{u}$. From Eq. (45) one obtains $w^{\mathrm{spin}}_{T}(\mathbf{s}, \mathbf{u})$ by including $\mathrm{Tr}(D_{\gamma} D^{\dagger}_{\gamma'})$ under the integral. Using that the durations of encounters are semiclassically large, compare (42), the result can be obtained in analogy to (34) by employing (27). The right-hand side then yields

$$w^{\mathrm{spin}}_{T}(\mathbf{s}, \mathbf{u}) \ \approx \ \frac{\left(T - \sum_{\alpha=1}^{V} l_{\alpha} t^{\alpha}_{enc}\right)^L}{\Sigma(E)^{L-V} \prod_{\alpha=1}^{V} t^{\alpha}_{enc} L!} M_{\gamma\gamma'} \ , \tag{46}$$

i.e. a factorization into the spin-independent part identical to $w_T(\mathbf{s}, \mathbf{u})$ and a spin contribution

$$\begin{aligned} M_{\gamma\gamma'} \ := \ & \int_{SU(2)} \cdots \int_{SU(2)} dg_L \ldots dg_2 \\ & \times \mathrm{Tr}\left(\pi_s\left(g_L \ldots g_2 g^{\eta_2 \dagger}_{k_2} \ldots g^{\eta_L \dagger}_{k_L}\right)\right) \ . \end{aligned} \tag{47}$$

In order to calculate (47) we follow the method developed in Refs. 20,21 for the spectral form factor of quantum graphs with spin-orbit interaction. In analogy to Theorem 6.1 of Ref. 21 we find in the present context that

$$M_{\gamma\gamma'} = (2s+1) \left(\frac{(-1)^{2s}}{2s+1}\right)^{L-V} \ . \tag{48}$$

This will be proven in the appendix. We stress that this spin contribution, apart from the spin quantum number, only depends on $L-V$.

The quantity (44) can now be calculated in analogy to the case without spin[9]. Starting from equation (43), one employs

the expressions for ΔS from (41) and for $w_T^{\rm spin}(\mathbf{s},\mathbf{u})$, the sum rule from Ref. 8 and the survival probability $\rho(t)$, modified by replacing t with $(t-\sum_{\alpha=1}^{V}(l_\alpha-1)t_{enc}^{\alpha})$ as in Ref. 9. This yields

$$\begin{aligned} T^{nd}_{a_2,a_1'} &\approx \left\langle \frac{(2s+1)\hbar}{mA}\sum_{\vec v}N(\vec v)\left(\prod_{i=1}^{L+1}\int_0^\infty dt_i \exp\left(-\frac{t_i}{\tau}\right)\right)\int_{-c}^{c}\cdots\int_{-c}^{c}\frac{d^{L-V}u\,d^{L-V}s}{(\Sigma(E))^{L-V}}\prod_{\alpha=1}^{V}\frac{\exp\left(-\frac{t_{enc}^{\alpha}}{\tau}+\frac{i}{\hbar}\Delta S\right)}{t_{enc}^{\alpha}}\right\rangle_{\Delta E}\left(\frac{(-1)^{2s}}{2s+1}\right)^{L-V} \\ &\approx \frac{(2s+1)}{N_1+N_2}\sum_{k=1}^{\infty}\left(\frac{1}{N_1+N_2}\right)^k\left(\frac{(-1)^{2s}}{2s+1}\right)^k\sum_{\vec v,L-V=k}(-1)^V N(\vec v)\,. \end{aligned} \tag{49}$$

The integrals over $\mathbf{s}$ and $\mathbf{u}$ were calculated in Ref. 9, and the sum over $\vec v$ can be carried out with the recursion formula[9]

$$\sum_{\vec v,L-V=k}(-1)^V N(\vec v)=\left(1-\frac{2}{\beta}\right)^k\,, \tag{50}$$

where $\beta=1$, if time reversal symmetry is present and $\beta=2$, if time reversal symmetry is broken.

Finally, using these results in the case of time-reversal invariance, we obtain for the full transmission matrix, including also the diagonal part,

$$T^{\rm nd}_{a_2,a_1'}+\frac{2s+1}{N_1+N_2}\approx\frac{(2s+1)^2}{(2s+1)(N_1+N_2)-1}\,, \tag{51}$$

in the case of half-integer s, and

$$T^{\rm nd}_{a_2,a_1'}+\frac{2s+1}{N_1+N_2}\approx\frac{(2s+1)^2}{(2s+1)(N_1+N_2)+1}\,, \tag{52}$$

if s is integer. For $s=1/2$ the result (51) is identical with the one obtained using Random Matrix Theory, in the circular symplectic ensemble[1].

These findings can now be compared with the respective results when time-reversal is absent, thus revealing the behavior of the transmission under a breaking of time-reversal by, e.g., turning on a magnetic field. In that case $\beta=2$ so that the term (50) vanishes, implying via (49) that only the diagonal contribution survives. The difference $\Delta\mathcal{T}=\mathcal{T}^{(\beta=1)}-\mathcal{T}^{(\beta=2)}$ of the transmission coefficients therefore is

$$\Delta\mathcal{T}\approx\frac{N_1N_2(2s+1)}{(N_1+N_2)\left((2s+1)(N_1+N_2)-1\right)}\,, \tag{53}$$

in the case of half-integer s, and

$$\Delta\mathcal{T}\approx\frac{-N_1N_2(2s+1)}{(N_1+N_2)\left((2s+1)(N_1+N_2)+1\right)}\,, \tag{54}$$

if s is integer. From these expressions one immediately concludes that the transmission (i.e., conductivity) is enhanced at zero magnetic field (when time reversal symmetry is restored), if the spin is half-integer; thus weak anti-localization occurs. To the contrary, integer spin would lead to weak localization. The latter had previously been obtained in semiclassical studies where the spin had been neglected[8]. The only semiclassical derivation of weak anti-localization so far[17], however, was restricted to the one-loop contribution and employed asymptotics for large N_1,N_2.

IV. SHOT NOISE

The techniques developed above can be applied to a number of further problems arising in the context of ballistic transport through chaotic mesoscopic cavities. As a first example we consider shot noise. To this end one needs to compute the energy-averaged Fano-factor F, defined as

$$F:=\frac{\left\langle {\rm Tr}(TT^\dagger-TT^\dagger TT^\dagger)\right\rangle_{\Delta E}}{\left\langle {\rm Tr}(TT^\dagger)\right\rangle_{\Delta E}}\,, \tag{55}$$

in terms of the transmission matrix $T=S^{21}$. The denominator has been dealt with above, and the spin-independent contribution to

$${\rm Tr}\left(TT^\dagger TT^\dagger\right) \tag{56}$$

was calculated semiclassically in Ref. 14. We are hence left with the task of determining the spin contribution to (56). Referring to the semiclassical representation (17) one immediately realizes that a four-fold sum over classical trajectories emerges. In addition to the case covered in Ref. 14 each term in this sum acquires an additional factor of

$${\rm Tr}\left(D_s^\dagger D_u D_v^\dagger D_w\right)\,, \tag{57}$$

in which the indices label the trajectories involved. The diagonal contribution to the four-fold sum occurs with $s=u$ and $v=w$, or with $s=w$ and $u=v$. In both cases unitarity implies

$${\rm Tr}\left(D_s^\dagger D_u D_v^\dagger D_w\right)=2s+1\,. \tag{58}$$

Beyond this one has to consider the encounter of four trajectories. For the first time this has been done in quantum graphs[13], and has later been extended in Ref. 14. Following the method of these papers, every trajectory consists of two parts, labeled by 1 and 2. Approximately, one then has $s_1=w_1$, $u_1=v_1$, $s_2=u_2$ and $v_2=w_2$. Thus

$$\begin{aligned} {\rm Tr}\left(D_s^\dagger D_u D_v^\dagger D_w\right) &\approx {\rm Tr}\left(D_{s_1}^\dagger D_{s_2}^\dagger D_{s_2} D_{v_1} D_{v_1}^\dagger D_{v_2}^\dagger D_{v_2} D_{s_1}\right) \\ &= 2s+1\,. \end{aligned} \tag{59}$$

Following further the calculation of the Fano factor in Ref. 14, we obtain

$$F \approx \frac{N_1 N_2}{(N_1+N_2)^2} \;, \tag{60}$$

for $N_1, N_2 \gg 1$. This result coincides with the respective outcome of a random matrix calculation in the symplectic ensemble[1,16].

V. CONDUCTANCE FLUCTUATIONS

Universality of conductance fluctuations is often characterized in terms of the energy-averaged variance of $\mathrm{Tr}(TT^\dagger)$. Instead of this quantity, the energy-averaged covariance of $\mathrm{Tr}(R^n R^{n\dagger})$, where $n = 1,2$ labels the leads, can also be considered, see Ref. 11 for details. Our calculations are based on the first paper of Ref. 11, whose method can still be applied when the Ehrenfest time is much smaller than the dwell time; this condition is fulfilled in the semiclassical limit considered here.

The calculation of the variances again involves four-fold sums over trajectories, in which the spin contribution occurs in terms of the factors

$$\mathrm{Tr}\left(D_s D_u^\dagger\right) \mathrm{Tr}\left(D_v D_w^\dagger\right) \;. \tag{61}$$

Switching off the spin-orbit interaction while preserving the presence of spin s, one obtains

$$\mathrm{Tr}\left(D_s D_u^\dagger\right) \mathrm{Tr}\left(D_v D_w^\dagger\right) = (2s+1)^2 \;. \tag{62}$$

In the presence of spin-orbit interaction one must examine the trajectories involved more closely. Here we again consider the case $N_1, N_2 \gg 1$. The trajectories are divided into three parts labeled by 1, 2 and 3, and the relations $s_1 = u_1$, $s_2 = \overline{v}_2$, $s_3 = u_3$, $v_1 = w_1$, $u_2 = \overline{w}_2$, $v_3 = w_3$ or $s_1 = u_1$, $s_2 = v_2$, $s_3 = u_3$, $v_1 = w_1$, $u_2 = w_2$, $v_3 = w_3$ hold approximately. Here an over-bar indicates that these pieces are traversed in reverse direction. In the first case this yields

$$\mathrm{Tr}\left(D_s D_u^\dagger\right) \mathrm{Tr}\left(D_v D_w^\dagger\right) \approx \mathrm{Tr}\left(D_{s_2} D_{u_2}^\dagger\right) \mathrm{Tr}\left(D_{u_2} D_{s_2}^\dagger\right) \;, \tag{63}$$

whereas in the second case

$$\mathrm{Tr}\left(D_s D_u^\dagger\right) \mathrm{Tr}\left(D_v D_w^\dagger\right) \approx \mathrm{Tr}\left(D_{s_2} D_{u_2}^\dagger\right)^2 \;. \tag{64}$$

After an average over $\mathrm{SU}(2)$, very much alike in the main part of this work, we obtain for the first case[20,21]

$$\int_{SU(2)} \int_{SU(2)} dg_a dg_b \mathrm{Tr}\left(\pi_s\left(g_a g_b^\dagger\right)\right) \mathrm{Tr}\left(\pi_s\left(g_b g_a^\dagger\right)\right) = 1 \;, \tag{65}$$

and for the second case[18]

$$\int_{SU(2)} \int_{SU(2)} dg_a dg_b \left[\mathrm{Tr}\left(\pi_s\left(g_a g_b^\dagger\right)\right)\right]^2 = 1 \;. \tag{66}$$

We follow Ref. 11 further and finally observe that, with $N_1, N_2 \gg 1$, the energy-averaged variance of $\mathrm{Tr}(TT^\dagger)$ reads

$$\left\langle \mathrm{var}\left(\mathrm{Tr}\left(TT^\dagger\right)\right)\right\rangle_{\Delta E} \approx 2\,(2s+1)^2 \frac{(N_1N_2)^2}{(N_1+N_2)^4} \;, \tag{67}$$

when the spin-orbit interaction is switched off, and

$$\left\langle \mathrm{var}\left(\mathrm{Tr}\left(TT^\dagger\right)\right)\right\rangle_{\Delta E} \approx 2\,\frac{(N_1N_2)^2}{(N_1+N_2)^4} \tag{68}$$

in the presence of spin-orbit interaction. Again, this finding is in accordance with the respective result in the symplectic ensemble of RMT[1,16].

VI. SUMMARY AND CONCLUSIONS

We considered the semiclassical description of ballistic transport through chaotic mesoscopic cavities in the presence of spin-orbit interactions. Our focus was the calculation of transmission coefficients. Here the principal task was to verify the effect of weak anti-localization in the form predicted by RMT.

Working within the framework of the Landauer formalism, our starting point was a semiclassical representation of Green functions for Hamiltonians that contain a spin-orbit interaction. Transmission coefficients then require the evaluation of double sums over classical trajectories. The principal difficulty presented by such expressions is to get hold of the interferences thus occurring. This can be overcome successfully by exploiting the Sieber-Richter method, originally developed to perform analogous calculations in the context of spectral fluctuations in classically chaotic quantum systems.

We attacked the problem using the two established variants of the Sieber-Richter method: the configuration-space approach for the leading order, and the phase-space approach for the remaining contributions. In the first case a key input was a classical sum rule encoding an ergodic (and mixing) behavior of the combined classical spin-orbit dynamics. Essential to the success of the phase-space approach was a calculation of the spin contribution to pairs of classical trajectories that are grouped together pairwise according to the structure of their almost self-encounters. This led to the central result given in Eq. (48). The sign appearing points to the essential difference between the effects of half-integer spin as opposed to integer spin (including spin zero). This difference was then identified as responsible for weak anti-localization or localization, respectively, to occur. We finally showed how our approach generalizes to semiclassical descriptions of shot noise and of universal conductance fluctuations.

Appendix A: PROOF OF THE RELATION (48)

We will show the validity of Eq. (48) by induction with respect to the number n of 2-encounters of two trajectories

$\gamma \neq \gamma'$. The proof is based on the relations

$$\int_{SU(2)} dg \mathrm{Tr}\left(\pi_s(xgyg)\right) = \frac{(-1)^{2s}}{2s+1} \mathrm{Tr}\left(\pi_s(xy^{-1})\right) \qquad \text{(A1)}$$

and

$$\int_{SU(2)} \int_{SU(2)} dg\, dh \quad \mathrm{Tr}\left(\pi_s(gwh^{-1}xg^{-1}yhz)\right) = \frac{1}{(2s+1)^2} \mathrm{Tr}\left(\pi_s(yxwz)\right), \quad \text{(A2)}$$

valid for all $w,x,y,z \in SU(2)$. For finite groups analogous identities have been shown in Ref. 21; their proofs can be directly carried over to the present case.

We now proceed in three steps:

1. First consider the case $n = 0$, where $\gamma' = \gamma$. This also means $\eta_j = 1$ and $k_j = j$. Here we obtain

$$\begin{aligned} M_{\gamma\gamma} &= \int_{SU(2)} \cdots \int_{SU(2)} dg_L \ldots dg_2 \\ &\quad \times \mathrm{Tr}\left(\pi_s\left(g_L \ldots g_2 g_2^\dagger \ldots g_L^\dagger\right)\right) \\ &= 2s+1\,. \end{aligned} \qquad \text{(A3)}$$

2. We assume the validity of (48) for two trajectories $\gamma = (l_1,a,b,l_2,c,d,l_3)$ and $\gamma' = (l_4,a,b,l_5,c,d,l_6)$ as shown in Figure 3. Here l_j stands for stretches of the trajectories γ and γ' containing an unspecified number of 2-encounters. By assumption, the actual number of 2-encounters, where γ differs from γ' is n. We show now that the relation (48) is still valid, when we replace γ' with the trajectory $\gamma'' = \left(l_4,a,\bar{c},\bar{l}_5,\bar{b},d,l_6\right)$. Thus γ'' differs from γ in $n' = n+1$ 2-encounters. Then

$$\begin{aligned} M_{\gamma\gamma''} &= \int_{SU(2)} \cdots \int_{SU(2)} dg_a dg_b dg_c dg_d \ldots \mathrm{Tr}\left(\pi_s\left(g_{l_3} g_d g_c g_{l_2} g_b g_a g_{l_1} g^\dagger_{l_4} g^\dagger_a g_c g_{l_5} g_b g^\dagger_d g^\dagger_{l_6}\right)\right) \\ &= \int_{SU(2)} \cdots \int_{SU(2)} dg_x dg_y dg_z \ldots \mathrm{Tr}\left(\pi_s\left(g_{l_3} g_x g_{l_2} g_y g_{l_1} g^\dagger_{l_4} g^\dagger_y g_z g_{l_5} g_z g^\dagger_x g^\dagger_{l_6}\right)\right) \\ &= \frac{(-1)^{2s}}{2s+1} \int_{SU(2)} \cdots \int_{SU(2)} dg_x dg_y \ldots \mathrm{Tr}\left(\pi_s\left(g_{l_3} g_x g_{l_2} g_y g_{l_1} g^\dagger_{l_4} g^\dagger_y g^\dagger_{l_5} g^\dagger_x g^\dagger_{l_6}\right)\right) \\ &= \frac{(-1)^{2s}}{2s+1} \int_{SU(2)} \cdots \int_{SU(2)} dg_a dg_b dg_c dg_d \ldots \mathrm{Tr}\left(\pi_s\left(g_{l_3} g_d g_c g_{l_2} g_b g_a g_{l_1} g^\dagger_{l_4} g^\dagger_a g^\dagger_b g^\dagger_{l_5} g^\dagger_c g^\dagger_d g^\dagger_{l_6}\right)\right) \\ &= \frac{(-1)^{2s}}{2s+1} M_{\gamma\gamma'}\,. \end{aligned} \qquad \text{(A4)}$$

In the second step we substituted $g_d g_c = g_x$, $g_b g_c = g_z$ and $g_b g_a = g_y$, and in the third one we used Eq. (A1). In the fourth step we undid the substitution. This calculation proves that changing the number of 2-encounters, in which γ and γ' differ, by one indeed contributes a factor of $(-1)^{2s}/(2s+1)$.

3. We assume the validity of the relation (48) for the two trajectories $\gamma = (l_1,a_1,b_1,l_2,a_2,b_2,l_3,c_1,d_1,l_4,c_2,d_2,l_5)$ and $\gamma' = (l_6,a_1,b_1,l_7,a_2,b_2,l_8,c_1,d_1,l_9,c_2,d_2,l_{10})$ as shown in Figure 4. Again we assume that the number of 2-encounters, where γ differs from γ', is n. We then show that the relation (48) is unchanged under a replacement of γ' with the trajectory $\gamma'' = (l_6,a_1,d_1,l_9,c_2,b_2,l_8,c_1,b_1,l_7,a_2,d_2,l_{10})$. Notice that γ'' cannot be constructed by applying the procedure of 2. twice: here the stretches l_6, l_7 and l_9 of γ' are traversed in parallel direction, whereas in 2. the stretches l_4 and l_6 of γ' are traversed in anti-parallel direction. A calculation similar to (A4), with the substitutions $g_{d_j} g_{c_j} = g_{x_j}$, $g_{b_j} g^\dagger_{d_j} = g_{z_j}$, $g_{b_j} g_{a_j} = g_{y_j}$ $(j \in \{1,2\})$, then yields

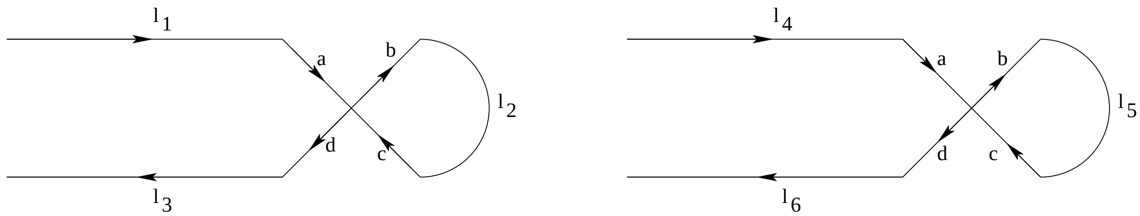

Figure 3: Sketches of the trajectories γ (left) and γ' (right) that are considered under 2.

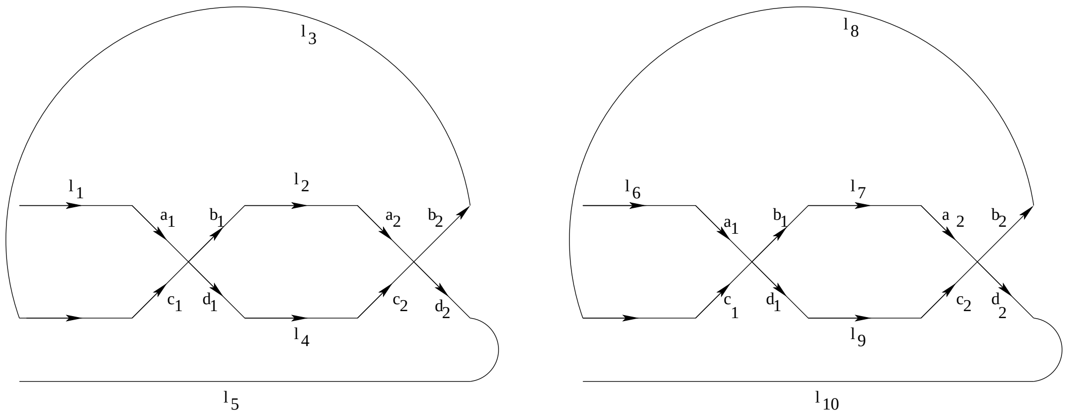

Figure 4: Sketches of the trajectories γ (left) and γ' (right) that are considered under 3.

$$
\begin{aligned}
M_{\gamma\gamma''} &= \int_{SU(2)} \cdots \int_{SU(2)} dg_{a_1} \ldots \mathrm{Tr}\left(\pi_s\left(g_{l_5} g_{d_2} g_{c_2} g_{l_4} g_{d_1} g_{c_1} g_{l_3} g_{b_2} g_{a_2} g_{l_2} g_{b_1} g_{a_1} g_{l_1} g^\dagger_{l_6} g^\dagger_{a_1} g^\dagger_{d_1} g^\dagger_{l_9} g^\dagger_{c_2} g^\dagger_{b_2} g^\dagger_{l_8} g^\dagger_{c_1} g^\dagger_{b_1} g^\dagger_{l_7} g^\dagger_{a_2} g^\dagger_{d_2} g^\dagger_{l_{10}}\right)\right) \\
&= \int_{SU(2)} \cdots \int_{SU(2)} dg_{x_1} \ldots \mathrm{Tr}\left(\pi_s\left(g_{l_5} g_{x_2} g_{l_4} g_{x_1} g_{l_3} g_{y_2} g_{l_2} g_{y_1} g_{l_1} g^\dagger_{l_6} g^\dagger_{y_1} g_{z_1} g^\dagger_{l_9} g^\dagger_{x_2} g^\dagger_{z_2} g^\dagger_{l_8} g^\dagger_{x_1} g^\dagger_{z_1} g^\dagger_{l_7} g^\dagger_{y_2} g_{z_2} g^\dagger_{l_{10}}\right)\right) \\
&= \frac{1}{(2s+1)^2} \int_{SU(2)} \cdots \int_{SU(2)} dg_{x_1} dg_{y_1} dg_{x_2} dg_{y_2} \ldots \mathrm{Tr}\left(\pi_s\left(g^\dagger_{l_7} g^\dagger_{y_2} g^\dagger_{l_8} g^\dagger_{x_1} g^\dagger_{l_9} g^\dagger_{x_2} g^\dagger_{l_{10}} g_{l_5} g_{x_2} g_{l_4} g_{x_1} g_{l_3} g_{y_2} g_{l_2} g_{y_1} g_{l_1} g^\dagger_{l_6} g^\dagger_{y_1}\right)\right) \\
&= \frac{1}{(2s+1)^2} M_{\gamma\gamma'} \ .
\end{aligned} \tag{A5}
$$

After these steps (48) follows by induction because every trajectory γ' can be constructed successively out of γ by using the procedures of 2. and 3. Every l-encounter that does not decompose into several encounters of a lower number of trajectories (see Figure 4 in Ref. 6 for an example) can be constructed from 2-encounters in $l-1$ steps. Every such step then brings out a factor of $(-1)^{2s}/(2s+1)$ in $M_{\gamma,\gamma'}$, when this is constructed from $M_{\gamma,\gamma} = 2s+1$. Thus, V encounters with altogether L stretches contribute a factor $\left((-1)^{2s}/(2s+1)\right)^{L-V}$, which completes the proof of (48).

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